Supersymmetric gauge theories in three dimensions

Bergshoeff, Eric A.; Hohm, Olaf

Published in:
Fortschritte der physik-Progress of physics

DOI:
10.1002/prop.200900047

Fortschr. Phys. **57**, No. 5 – 7, 338 – 347 (2009) / **DOI** 10.1002/prop.200900047

Supersymmetric gauge theories in three dimensions

Eric A. Bergshoeff* and **Olaf Hohm****

Center for Theoretical Physics, University of Groningen, Nijenborg 4, 9747 AG Groningen, The Netherlands

Received 20 February 2009, accepted 27 February 2009
Published online 6 April 2009

Key words M theory, conformal field theory.
PACS 11.25.Yb, 11.25.Hf

We show how superconformal gaugings in three dimensions can be systematically constructed using the embedding tensor technique. These gaugings have been argued to describe the worldvolume theory of multiple M2-branes. Applying our technique we construct the most general superconformal gaugings with $\mathcal{N} = 5, 6$ and 8 supersymmetry. In the case of $\mathcal{N} = 5$ supersymmetry we find three exceptional gaugings. We briefly discuss new developments concerning the massive deformations of the superconformal gauge theories.

1 Introduction

Already back in 1987 the covariant (and kappa-symmetric) action of a single M2-brane has been constructed [1]. The basic variables describing the worldvolume theory are the eleven-dimensional embedding coordinates X^μ and their fermionic partners θ^α. The latter variables are spinors from the eleven-dimensional spacetime point of view but scalars from the worldvolume point of view. After imposing the lightcone gauge one is left with 8 bosonic degrees of freedom X^I $(I = 1, \cdots, 8)$ and the same number of fermionic degrees of freedom which are now fermions from the worldvolume point of view. Dualizing one of the eight transverse scalars into a worldvolume vector, thereby breaking the transverse $\mathrm{SO}(8)$ symmetry to $\mathrm{SO}(7)$, one obtains the Born-Infeld action of a single D2-brane.

In the case of D2-branes it is well-known how to describe a set of multiple, overlapping, D2-branes. The corresponding worldvolume theory is given by a $\mathrm{U}(N)$ Yang-Mills theory. On the other hand, it is known that the strong coupling limit of this Yang-Mills theory should lead to M-theory with multiple M2-branes. Since the strong coupling limit of three-dimensional Yang-Mills is given by a conformal-invariant fixed point one might hope that multiple M2-branes are described by a worldvolume theory with a superconformal gauging. It is non-trivial to construct such a superconformal gauging but recently, using the earlier observations of [2], it has been shown how this can be done for the relevant case of $\mathcal{N} = 8$ supersymmetry [3–7]. This is the so-called BLG model.

In this talk we will show how the same superconformal gaugings can be understood and constructed by using the so-called embedding tensor technique. Originally, this technique was developed to construct gauge theories with local supersymmetry, i.e.gauged supergravities. Here we will discuss the construction directly in the context of global supersymmetry. Alternatively, one could first consider gauged supergravity and then take the conformal limit to global supersymmetry. This approach is discussed in the talk by O. Hohm at the conference. The content of this talk is based on [8] and [9].

* Corresponding author E-mail: E.A.Bergshoeff@rug.nl., Phone: +00 31 (0)50 3634969, Fax: +00 31 (0)50 4947
** E-mail: O.Hohm@rug.nl.

2 The embedding tensor technique

Starting from an ungauged theory with global symmetry group $\hat{G}$ the gauging of a subgroup $\hat{G}_0 \subset \hat{G}$ can be achieved by applying the so-called embedding tensor technique [10–12]. In the case at hand the embedding tensor $\Theta_{\alpha\beta} = \Theta_{\beta\alpha}$ takes values in the symmetric product of the adjoint representation of $\hat{G}$:

$$\Theta \in (\mathrm{Adj}(\hat{G}) \otimes \mathrm{Adj}(\hat{G}))_{\mathrm{symm}}\,, \tag{1}$$

and relates gauge vectors to generators of $\hat{G}$. The associated transformations are then gauged due to the introduction of the embedding tensor in covariant derivatives which take the general form

$$D_\mu = \partial_\mu - A_\mu{}^\alpha\, \Theta_{\alpha\beta}\, t^\beta\,, \tag{2}$$

for some representation matrices t^β of of $\hat{G}$. Note that $X_\alpha = \Theta_{\alpha\beta} t^\beta$ denote the generators whose symmetries are being gauged. Thus, the embedding tensor determines which subgroup $\hat{G}_0$ of $\hat{G}$ is being gauged and which vectors are being used for this gauging. We have assumed that the vectors are in the adjoint representation of $\hat{G}$. The reason for this is that only then are we able to introduce the vectors in the Lagrangian via the following Chern-Simons term where the embedding tensor appears as a metric:

$$\mathcal{L}_{\mathrm{CS}} = \frac{1}{4}\,\varepsilon^{\mu\nu\rho}\, A_\mu^\alpha\, \Theta_{\alpha\beta}\Big(\partial_\nu A_\rho^\beta - \frac{1}{3}\,\Theta_{\gamma\epsilon}\, f^{\beta\epsilon}{}_\delta\, A_\nu^\gamma A_\rho^\delta\Big)\,. \tag{3}$$

Here $f^{\alpha\beta}{}_\gamma$ are the structure constants of $\hat{G}$. The nice thing about this Chern-Simons coupling is that it does not introduce new degrees of freedom for the vectors. Instead, the field equation for the vectors lead to a duality relation between the vectors and the scalars. Hence, all independent bosonic degrees of freedom are described by scalars X^{aI} $(a = 1, \cdots, N)$, as in the ungauged case. Here N is a number that is related to, but not necessarily equal to, the number of overlapping M2-branes.

It turns out that, after the replacement of ordinary derivatives by covariant derivatives, the Lagrangian can be completed to a supersymmetric and gauge-invariant one provided the embedding tensor satisfies a number of constraints. These constraints are either linear or quadratic in Θ and are called *linear* and *quadratic* constraints, respectively. The quadratic constraint follows from the requirement that the embedding tensor itself is invariant under the transformations that are gauged due to the introduction of Θ. This condition takes the same form for all values of $\mathcal{N}$:

$$Q_{\beta,\delta\epsilon} \equiv \Theta_{\alpha\beta}\Theta_{\gamma(\delta}\, f^{\alpha\gamma}{}_{\epsilon)} = 0\,. \tag{4}$$

In case the embedding tensor projects onto a semisimple subgroup of $\hat{G}$ and is expressed in terms of invariant tensors of that subgroup, the quadratic constraint (4) is automatically satisfied.

The linear constraints are case-dependent. We will discuss them now for $\mathcal{N} = 5, 6$ and 8 supersymmetry. For these cases the rigid symmetry group has the following product structure:

$$\hat{G} = \big(G \times H_{\mathrm{R}}\big) \ltimes \mathbb{R}^{8N}\,, \tag{5}$$

with G and the R-symmetry group H_{R} given in Table 1. It is well-known that in a globally supersymmetric gauge theory one cannot gauge a subgroup of the R-symmetry group H_{R} since that would require to gauge supersymmetry. We restrict to a gauging of a subgroup G_0 of G, i.e. we do not consider gauging translations. In other words, we take the embedding tensor to have only indices in G, i.e. α is an adjoint index of G. It turns out that this choice always leads to a conformal gauging. The index α can be replaced by a pair of indices a, b in the fundamental representation of G as follows:

$$\mathcal{N} = 8:\ \ \alpha \to [ab] \qquad \mathcal{N} = 6:\ \ \alpha \to {}_a{}^b \qquad \mathcal{N} = 5:\ \ \alpha \to (ab)\,. \tag{6}$$

Table 1 G and $H_{\rm R}$ for different numbers $\mathcal{N}$ of supersymmetry

$\mathcal{N}$	G	$H_{\rm R}$
8	$\mathrm{SO}(N)$	$\mathrm{SO}(8)$
6	$\mathrm{U}(N)$	$\mathrm{SO}(6) \sim \mathrm{SU}(4)$
5	$\mathrm{Sp}(N)$	$\mathrm{SO}(5) \sim \mathrm{Sp}(2)$

Imposing the linear constraints we find that the non-trivial components of the embedding tensor are given by the following Young tableaux [8]:

• $\mathcal{N} = 8$

The embedding tensor takes values in the following representation of $G = \mathrm{SO}(N)$:

$$\yng(1,1,1,1)\,, \tag{7}$$

and as a consequence is totally anti-symmetric

$$\Theta_{ab,cd} = \Theta_{[ab,cd]}\,. \tag{8}$$

With the $\mathrm{SO}(N)$ structure constants

$$f^{ab,cd,}{}_{ef} = -2\delta^{[a}{}_{[e}\delta^{b][c}\delta^{d]}{}_{f]}\,, \tag{9}$$

the quadratic constraint (4) takes the form

$$\Theta_{ab,e}{}^{g}\Theta_{cd,gf} + \Theta_{ab,c}{}^{g}\Theta_{ef,gd} - \Theta_{ab,f}{}^{g}\Theta_{cd,ge} - \Theta_{ab,d}{}^{g}\Theta_{ef,gc} = 0\,. \tag{10}$$

• $\mathcal{N} = 6$

The embedding tensor takes values in the following representations of $G = \mathrm{U}(N)$:

$$1 \oplus \yng(1)\,\overline{\yng(1)} \oplus \yng(1,1)\,\overline{\yng(1,1)}\,, \tag{11}$$

and therefore is anti-symmetric in its two pairs of indices:

$$\Theta_{a}{}^{b}{}_{,c}{}^{d} = \Theta_{[a}{}^{b}{}_{,c]}{}^{d}\,. \tag{12}$$

With the $\mathrm{U}(N)$ structure constants

$$f_{a}{}^{b}{}_{,\,c}{}^{d}{}_{,}{}^{f}{}_{e} = i\left(\delta_{c}{}^{b}\delta_{a}{}^{f}\delta_{e}{}^{d} - \delta_{a}{}^{d}\delta_{c}{}^{f}\delta_{e}{}^{b}\right)\,, \tag{13}$$

the quadratic constraint (4) takes the form

$$\Theta_{c}{}^{g}{}_{,e}{}^{f}\,\Theta_{g}{}^{d}{}_{,a}{}^{b} - \Theta_{g}{}^{d}{}_{,e}{}^{f}\,\Theta_{c}{}^{g}{}_{,a}{}^{b} + \Theta_{a}{}^{g}{}_{,e}{}^{f}\,\Theta_{g}{}^{b}{}_{,c}{}^{d} - \Theta_{g}{}^{b}{}_{,e}{}^{f}\,\Theta_{a}{}^{g}{}_{,c}{}^{d} = 0\,. \tag{14}$$

• $\mathcal{N} = 5$

The embedding tensor takes values in the following representations of $G = \mathrm{Sp}(N)$:

$$1 \oplus \yng(1,1) \oplus \yng(2,2) \tag{15}$$

and hence satisfies the linear constraint

$$\Theta_{(ab,cd)} = 0\,. \tag{16}$$

With the $\mathrm{Sp}(N)$ structure constants

$$f^{ab,cd,}{}_{ef} = -2\delta^{(a}{}_{(e}\Omega^{b)(c}\delta^{d)}{}_{f)}\,, \tag{17}$$

the quadratic constraint (4) takes the form

$$\Omega^{gh}\left(\Theta_{ab,eg}\Theta_{hf,cd} + \Theta_{ab,fg}\Theta_{he,cd} + \Theta_{ab,cg}\Theta_{hd,ef} + \Theta_{ab,dg}\Theta_{hc,ef}\right) = 0\,. \tag{18}$$

3 Solving the constraints

There are in general two strategies to solve the set of linear and quadratic constraints on the embedding tensor. Either one starts from an embedding tensor which projects onto a given subgroup by means of invariant tensors such that the quadratic constraint is automatically satisfied. In this case, the linear constraint becomes a non-trivial identity which decides if the gauging is a viable one. Alternatively, one may start from the general solution of the linear constraint which can directly be expressed in terms of the proper subrepresentations. Then, the quadratic constraint becomes a non-trivial identity which selects the proper gaugings. In both approaches, one set of constraints is trivially satisfied while the other one becomes a non-trivial identity. Both approaches have been pursued in the literature and depending on the point of view, the remaining constraint (which is the linear one in the superpotential formalism of [13, 14] and the quadratic one in the 3-algebra formalism of [4, 6, 15]) has been referred to as *fundamental identity*, respectively.

We will choose the first approach. Our starting point is an embedding tensor Θ that has only directions in G. Our task is now to construct an expression for Θ using only projectors onto subgroups G_0 of G which are made out of invariant tensors of G_0. In this way we have automatically solved the quadratic constraint. This leaves us with some undetermined coefficients which are in a second stage determined by solving the linear constraints. We first discuss how to construct the projectors onto G_0.

For the classical Lie groups we will use the standard invariant tensors δ_{ab} (orthogonal groups), $\delta_a{}^b$ (unitary groups) and both $\delta_a{}^b$ and $\Omega_{ab} = -\Omega_{ba}$ (symplectic groups). Here δ denotes the Kronecker delta and Ω the anti-symmetric symplectic tensor with inverse tensor Ω^{ab}, i.e. $\Omega_{ac}\,\Omega^{bc} = \delta_a{}^b$. Besides these tensors we will also use the following special invariant tensors in the case of $\mathrm{SO}(4)$, $\mathrm{SO}(7)$ and G_2:

$$\mathrm{SO}(4)\ :\ \epsilon_{abcd}\,, \qquad G_2\ :\ C_{abcd}\,, \qquad \mathrm{SO}(7)\ :\ \Gamma^{mn}_{ab}\Gamma^{mn}_{cd}\,. \tag{19}$$

Here $C_{abcd} = C_{[abcd]}$ is the unique 4-index invariant tensor of G_2 and the Γ^m are $\mathrm{SO}(7)$ Gamma-matrices. The index a refers to the 4-dimensional fundamental representation of $\mathrm{SO}(4)$, the 7-dimensional fundamental representation of G_2 and the 8-dimensional spinor representation of $\mathrm{SO}(7)$, respectively. We now wish to construct, using the above invariant tensors, the operators that project the Lie algebra generators of the global symmetry group G onto the generators of the subgroup G_0 which is gauged. Furthermore we will also need the operators that project onto the singlet representation. These operators will be the building blocks from which we will construct the embedding tensor. In the case of the classical orthogonal, unitary and symplectic groups these building blocks are given by:

$$\begin{aligned}
&\mathrm{SO}(N)\ \text{singlet:}\quad \delta_{ab}\,\delta_{cd}\,, && \mathrm{SO}(N)\ \text{adjoint:}\ \delta_{c[a}\,\delta_{b]d}\,,\\
&\mathrm{SU}(N)\ \text{singlet:}\quad \delta_a{}^b\,\delta_c{}^d\,, && \mathrm{SU}(N)\ \text{adjoint:}\ (\delta_c{}^b\,\delta_a{}^d - \frac{1}{N}\delta_a{}^b\,\delta_c{}^d)\,,\\
&\mathrm{Sp}(N)\ \text{singlet:}\quad \Omega_{ab}\,\Omega_{cd}\,, && \mathrm{Sp}(N)\ \text{adjoint:}\ \Omega_{c(a}\,\Omega_{b)d}\,.
\end{aligned} \tag{20}$$

For $\mathrm{SO}(4)$, G_2 and $\mathrm{SO}(7)$ there are additional operators that projects onto the adjoint representation given by

$$\mathrm{SO}(4)\ \text{adjoint:}\ \epsilon_{abcd}\,, \qquad G_2\ \text{adjoint:}\ \big(\delta_{a[c}\delta_{d]b} + \tfrac{1}{4}C_{abcd}\big)\,, \qquad \mathrm{SO}(7)\ \text{adjoint:}\ \Gamma^{mn}_{ab}\Gamma^{mn}_{cd}\,. \tag{21}$$

To allow for gauge groups G_0 with a product structure we will need to split the index a according to a pair of indices $(i, \bar{i})$:

$$a \to (i, \bar{i}) \qquad \text{with} \qquad i = 1, \dots, m\,;\ \bar{i} = 1, \dots, n\,, \tag{22}$$

corresponding to a bi-fundamental representation. These cases will be referred to as matrix models. Clearly, $n = 1$ is a special case for which the matrix reduces to a vector, and the indices a and i coincide. Another possibility is to split up a in a sum of indices, but this will only lead to direct sums of theories. Finally, one could of course consider more complicated solutions which make use of splittings like $a \to (i, \bar{i}, \tilde{i})$ but in practice such solutions do not occur.

Having satisfied the quadratic constraint by employing the above building blocks, we now discuss the solution of the linear constraint for the different cases with decreasing number of supersymmetry separately.

• $\mathcal{N} = 8$

In this case the embedding tensor contains only one irreducible component under $\mathrm{SO}(N)$, that is the 4-index anti-symmetric tensor $\Theta_{ab,cd} = \Theta_{[ab,cd]}$. Therefore, one cannot use the Kronecker delta δ_{ab} to make an expression for Θ.

One possibility is to make use of the special operator given in (21) and write

$$\Theta_{ab,cd} = g\,\epsilon_{abcd}\,, \tag{23}$$

for arbitrary coupling constant g. This restricts to $N = 4$ and $\mathrm{SO}(4)$ gauging.

Another possibility is to consider a symplectic gauging and to construct an invariant embedding tensor of the form $\Theta_{abcd} \sim \Omega_{[ab}\Omega_{cd]}$. However, according to Eq. (20) this is not an $\mathrm{Sp}(N)$ projection operator. Therefore, the quadratic constraint will not be satisfied and one cannot consider this possibility. Note that the gauging of a $G_2 \subset \mathrm{SO}(7)$ subgroup is neither possible because, although C_{abcd} is totally anti-symmetric and hence satisfies the linear constraint (8), the combination $\delta_{a[c}\delta_{d]b} + \frac{1}{4}\,C_{abcd}$, which is needed for closure, see Eq. (21), does *not* satisfy the $\mathcal{N} = 8$ linear constraint [19]. We conclude that, for $\mathcal{N} = 8$ one can only gauge $SO(4)$ or multiple copies thereof.

• $\mathcal{N} = 6$

In this case we are dealing with an embedding tensor $\Theta_a{}^b{}_{,c}{}^d$ that satisfies the linear constraint (12). Since the embedding tensor has both upper and lower indices we can use the invariant Kronecker delta $\delta_a{}^b$ to make expressions for Θ. This does not restrict to particular values of N. That is the basic reason why for $\mathcal{N} = 6$ one can obtain gaugings for arbitrary N [16].

The easiest way to find a solution that satisfies the linear constraint is to take

$$\Theta_a{}^b{}_{,c}{}^d = g\,\delta_{[a}{}^{[d}\,\delta_{c]}{}^{b]} \;=\; \frac{g}{2}\left(\delta_c{}^b\,\delta_a{}^d - \frac{1}{N}\,\delta_a{}^b\,\delta_c{}^d\right) \;-\; \frac{(N-1)g}{N\quad 2}\,\delta_a{}^b\,\delta_c{}^d\,, \tag{24}$$

for arbitrary coupling constant g. Note that the singlet operator becomes a $\mathrm{U}(1)$ projection operator. For $N > 1$ this picks out all generators of $\mathrm{U}(N)$ and leads to a gauging of the full $\mathrm{U}(N)$ group. Note that, in order to satisfy the linear constraint (12), we must take a specific combination of the $\mathrm{SU}(N)$ and $\mathrm{U}(1)$ operators.

We next consider a matrix model describing the embedding $\mathrm{SU}(m) \times \mathrm{SU}(n) \subset \mathrm{U}(N = mn)$ such that the scalars transform in the bi-fundamental representation (m, n). We furthermore allow for possible $\mathrm{U}(1)$ factors. We first try an embedding tensor that contains products of adjoints with singlets. However, one finds that one can not satisfy the linear constraint (12) with this Ansatz. For this we need to add a common $\mathrm{U}(1)$ factor that acts on both factors. We thus obtain

$$\begin{aligned}\Theta_{(i,\bar{i})}{}^{(k,\bar{k})}{}_{,\,(j,\bar{j})}{}^{(l,\bar{l})} = g\,\delta_{\bar{i}}{}^{\bar{k}}\delta_{\bar{j}}{}^{\bar{l}}\left(\delta_i{}^l\delta_j{}^k - \frac{1}{m}\delta_i{}^k\delta_j{}^l\right) - g\delta_j{}^l\delta_i{}^k\left(\delta_{\bar{j}}{}^{\bar{k}}\delta_{\bar{i}}{}^{\bar{l}} - \frac{1}{n}\delta_{\bar{i}}{}^{\bar{k}}\delta_{\bar{j}}{}^{\bar{l}}\right)\\ -\frac{(m-n)}{mn}g\,\delta_i{}^k\delta_j{}^l\delta_{\bar{i}}{}^{\bar{k}}\delta_{\bar{j}}{}^{\bar{l}}\,,\end{aligned} \tag{25}$$

for arbitrary coupling constant g. We deduce that the unitary matrix model desribes a $\mathrm{SU}(m) \times \mathrm{SU}(n) \times \mathrm{U}(1)$ gauging, corresponding to the $\mathrm{U}(m|n)$ model of [14]. For $m = n$, in which case the $\mathrm{U}(1)$ factor vanishes [17], this is the ABJM model of [18] .

Finally, we consider symplectic gaugings. Note that we can now raise and lower indices using the symplectic tensor. We first try an embedding tensor that only contains the adjoint of $\mathrm{Sp}(N)$. However, this does not satisfy the linear constraint (12) and we must add an additional $\mathrm{U}(1)$ factor (denoting the coupling contant by g):

$$\Theta_{ab,cd} \;=\; g\,\Omega_{ab}\Omega_{cd} \;-\; g\left(\Omega_{ca}\Omega_{bd} \;+\; \Omega_{cb}\Omega_{ad}\right), \tag{26}$$

where the first term on the right-hand-side corresponds to the $\mathrm{U}(1)$ gauging and where the term between round brackets corresponds to the $\mathrm{Sp}(N)$ gauging. This is precisely the so-called $\mathrm{OSp}(2|N)$ model of [14].

In summary, for $\mathcal{N} = 6$ there is a matrix model with $\mathrm{SU}(m) \times \mathrm{SU}(n) \times \mathrm{U}(1)$ gauging where the $\mathrm{U}(1)$ factor drops out if $m = n$ and a vector model with $\mathrm{U}(1) \times \mathrm{Sp}(N)$ gauging. The unitary matrix model gives rise, for $n = 1$, to a vector model with $\mathrm{U}(m)$ gauging. By taking multiple copies thereof one obtains vector models with $\mathrm{U}(m_1) \times \mathrm{U}(m_2) \times \cdots$ gauging, where $m_1 + m_2 + \ldots = N$.

• $\mathcal{N} \;=\; 5$

The global symmetry group for $\mathcal{N} = 5$ is $\mathrm{Sp}(N)$. We first try to gauge the full $\mathrm{Sp}(N)$ using the Ansatz

$$\Theta_{ab,cd} \;=\; g\;\Omega_{c(a}\,\Omega_{b)d}\,, \tag{27}$$

with arbitrary coupling constant g. This indeed solves the linear constraint (16) and leads to a vector model with $\mathrm{Sp}(N)$ gauging. Similarly, one can take multiple copies thereof with $\mathrm{Sp}(N_1) \times \mathrm{Sp}(N_2) \times \cdots$ gauging with $N_1 + N_2 + \ldots = N$.

It turns out that the vector model is a special case of a matrix model with $\mathrm{SO}(m) \times \mathrm{Sp}(n)$ gauging. The corresponding embedding tensor solving the linear constraint is given by

$$\Theta_{(i\bar{i})(j\bar{j}),(k\bar{k})(l\bar{l})} \;=\; g\left(\delta_{k[i}\;\delta_{j]l}\;\Omega_{\bar{i}\bar{j}}\;\Omega_{\bar{k}\bar{l}} + \;\delta_{ij}\;\delta_{kl}\;\Omega_{\bar{k}(\bar{i}}\;\Omega_{\bar{j})\bar{l}}\right)\;, \tag{28}$$

for arbitrary coupling constant g. This is precisely the so-called $\mathrm{OSp}(m|n)$ model of [14]. Note that the relative strength between the $\mathrm{SO}(m)$ and $\mathrm{Sp}(n)$ terms is fixed by the linear constraint (16). Matrix models with $\mathrm{SO}(m) \times \mathrm{SO}(n)$ or $\mathrm{Sp}(m) \times \mathrm{Sp}(n)$ gauging cannot be constructed simply because one cannot embed these into $\mathrm{Sp}(N)$.

Inspired by the connection with Lie superalgebras [13] we have also found three exceptional solutions. These solutions make use of the three special projectors defined in (21). The first one is based on the Lie superalgebra $F(4)$. The embedding tensor reads (where $a, b, ..$ refer to the spinor representation **8** of $SO(7)$ and $\alpha, \beta, ..$ denote an $SL(2)$ doublet)

$$\Theta_{a\alpha\,b\beta,c\gamma\,d\delta} = \Gamma^{mn}_{ab}\Gamma^{mn}_{cd}\,\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} + 12\delta_{ab}\delta_{cd}\epsilon_{\gamma(\alpha}\epsilon_{\beta)\delta}\,. \tag{29}$$

This gives rise to a gauging of $SO(7) \times SL(2)$.

The second possibility corresponds to the Lie superalgebra $G(3)$. The embedding tensor is given by

$$\Theta_{i\alpha\,j\beta,k\gamma\,l\delta} = (\delta_{i[k}\delta_{l]j} + \frac{1}{4}\,C_{ijkl})\,\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} + \delta_{ij}\delta_{kl}\epsilon_{\gamma(\alpha}\epsilon_{\beta)\delta}\,, \tag{30}$$

where $i, j, ..$ refer to the fundamental representation **7** of G_2 and $\alpha, \beta, ..$ denote an $SL(2)$ doublet. This leads to a $G_2 \times SL(2)$ gauge group.

Finally, the Lie superalgebra $\mathrm{OSp}(4|2;\alpha)$ gives a deformation of the $\mathrm{SO}(4) \times \mathrm{Sp}(2)$ gauging with embedding tensor

$$\Theta_{i\alpha\,j\beta,k\gamma\,l\delta} = (\delta_{i[k}\delta_{l]j} + \gamma/2\;\epsilon_{ijkl})\,\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} + \delta_{ij}\delta_{kl}\epsilon_{\gamma(\alpha}\epsilon_{\beta)\delta} \tag{31}$$

with $i, j = 1, .., 4$ of $SO(4)$ and $\alpha, \beta = 1, 2$ of $SL(2)$. This example also corresponds to a gauging of $\mathrm{SO}(4) \times \mathrm{Sp}(2)$, but where the two $\mathrm{SU}(2)$ coupling constants do not coincide when $\gamma \neq 0$. Here α is proportional to $(1 + \gamma)/(1 - \gamma)$.

This finishes our discussion of the $\mathcal{N} = 5, 6$ and 8 superconformal gaugings. We have summarized the different gauge groups in Table 2.

$\mathcal{N}$	gauge group $G_0 \subset G$
8	$\mathrm{SO}(4)$
6	$\mathrm{SU}(m) \times \mathrm{SU}(n) \times \mathrm{U}(1)$ $(m \neq n)$
6	$\mathrm{SU}(m) \times \mathrm{SU}(m)$
6	$\mathrm{SO}(2) \times \mathrm{Sp}(n)$
5	$\mathrm{SO}(m) \times \mathrm{Sp}(n)$
5	$\mathrm{SO}(7) \times \mathrm{SL}(2)$
5	$\mathrm{G}_2 \times \mathrm{SL}(2)$
5	$\mathrm{SO}(4) \times \mathrm{Sp}(2)$

Table 2 Possible gauge groups for different numbers $\mathcal{N}$ of supersymmetry

4 New developments

It turns out that the massive deformations constructed in [20,21] are based on the existence in three dimensions of superalgebras with non-central terms [22]. The possibility of a non-centrally extended superalgebra arises for $\mathcal{N} \geq 4$ supersymmetry. For $\mathcal{N} = 4$, the super-Poincaré algebra can be extended by the following non-central charges:

$$\{Q^i_\alpha, Q^j_\beta\} \; = \; 2\,(\gamma^\mu C)_{\alpha\beta}\, P_\mu \delta^{ij} + 2mC_{\alpha\beta}\varepsilon^{ijkl} M_{kl} \; , \tag{32}$$

where M_{ij} denote the $\mathrm{SO}(4)$ R-symmetry generators. In particular, they do not commute, but instead satisfy the standard relations

$$\begin{aligned} [M_{ij}, M_{kl}] \; &= \; -2\left(\delta_{k[i} M_{j]l} - \delta_{l[i} M_{j]k}\right) \; , \\ [M^{ij}, Q^k_\alpha] \; &= \; 2\delta^{k[i} Q^{j]}_\alpha \; . \end{aligned} \tag{33}$$

Algebras of this type also appear in the context of AdS supergroups, where the supercharges generically close into the R-symmetry group. The peculiar property here, however, is that this represents a consistent algebra for Poincaré supersymmetry, i.e., despite the commuting translations, the particular choice (32) containing an $\mathrm{SO}(4)$ Levi-Civita symbol satisfies the super-Jacobi identities.

This non-central extension is also possible for $\mathcal{N} > 4$. In the case of $\mathcal{N}$ being k multiples of 4 [1]. the $\mathrm{SO}(\mathcal{N})$ R-symmetry group will be broken to $\mathrm{SO}(4)^k$. For instance, in the case of $\mathcal{N} = 8$ supersymmetry the non-central charges occur at the right-hand-side of the $\{Q, Q\}$ anti-commutator as indicated schematically as follows:

$$\{Q, Q\} \sim \begin{pmatrix} \mathrm{non-central\ charge} & 0 \\ 0 & \mathrm{non-central\ charge} \end{pmatrix} . \tag{34}$$

This breaks the R-symmetry according to

$$\mathrm{SO}(8) \; \to \; \mathrm{SO}(4) \; \times \; \mathrm{SO}(4) \, . \tag{35}$$

The reprensention theory of these non-centrally extended superalgebras has been recently considered in [9]. We consider here only the case of $\mathcal{N} = 4$ supersymmetry. In this case the oscillator algebra that follows from the supersymmetry algebra reads $(i = 1, \cdots, 4)$

$$\{a^i, (a^j)^\dagger\} \; = \; M\delta^{ij} + m\varepsilon^{ijkl} M_{kl} \; . \tag{36}$$

[1] The case of $\mathcal{N} = 5, 6$ supersymmetry was considered in [14]. For these cases the superalgebra contains both the non-central charges considered here as well as conventional central charges.

It turns out to be convenient to construct the representations using SU(2) spinor indices via the isomorphism SO$(4) \cong$ SU$(2)_L \times$ SU$(2)_R$. Specifically, the oscillators are bispinors

$$a_{a\dot{a}} = \Gamma^i_{a\dot{a}}\, a^i \,, \tag{37}$$

where $\Gamma^i_{a\dot{a}}$ are SO(4) gamma matrices and we use undotted and dotted indices for SU$(2)_L$ and SU$(2)_R$, respectively. The SO(4) generators decompose accordingly into the symmetric SU$(2)_{L,R}$ generators M^{ab} and $M^{\dot{a}\dot{b}}$. Using this notation the algebra (36) reads

$$\begin{aligned} \{a_{a\dot{a}}, a^\dagger_{b\dot{b}}\} \;&=\; -2M\varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}} - 4m\left(\varepsilon_{\dot{a}\dot{b}}M_{ab} - \varepsilon_{ab}M_{\dot{a}\dot{b}}\right) \;, \\ \{a_{a\dot{a}}, a_{b\dot{b}}\} \;&=\; \{a^\dagger_{a\dot{a}}, a^\dagger_{b\dot{b}}\} \;=\; 0 \;. \end{aligned} \tag{38}$$

We note that the two SU(2) factors enter with a relative minus sign, which is due to their respective self-duality and anti-self-duality. In addition, the supercharges act as raising and lowering operators for the SU(2) quantum numbers. To be more precise, if one writes the spinor indices as $a = (+,-)$ and $\dot{a} = (\dot{+},\dot{-})$, then an undotted or dotted '+' index indicates that the SU$(2)_{L,R}$ spin quantum number is increased by $\frac{1}{2}$, while a '–' index indicates that is is decreased by $\frac{1}{2}$. Moreover, M_{+-} corresponds to the J_3 operator and thus measures the quantum number.

In order to construct shortened supermultiplets we must impose a generalized BPS condition. To see how this works, let us consider the bracket

$$\{a_{+\dot{+}}, (a_{+\dot{+}})^\dagger\} \;=\; -\{a_{+\dot{+}}, a^\dagger_{-\dot{-}}\} \;=\; 2M - 4m(J^L_3 - J^R_3) \;, \tag{39}$$

where we used $(a_{+\dot{+}})^\dagger = -\varepsilon^{+-}\varepsilon^{\dot{+}\dot{-}}a^\dagger_{-\dot{-}}$. In case the BPS-like condition $M = 2m(\ell_L - \ell_R)$ is satisfied, positivity of the Hilbert space implies that $a^\dagger_{-\dot{-}}$ is deactivated. Similarly, one derives from (38) that each of the four possible raising operators is deactivated provided the corresponding BPS condition is satisfied:

$$\begin{aligned} a^\dagger_{+\dot{+}} \;:& \qquad M = -2m(\ell_L - \ell_R) \;, \\ a^\dagger_{+\dot{-}} \;:& \qquad M = -2m(\ell_L + \ell_R) \;, \\ a^\dagger_{-\dot{+}} \;:& \qquad M = 2m(\ell_L + \ell_R) \;, \\ a^\dagger_{-\dot{-}} \;:& \qquad M = 2m(\ell_L - \ell_R) \;. \end{aligned} \tag{40}$$

Note that, in contrast to ordinary BPS multiplets, different sets of supercharges become trivial, depending on which states they act.

To construct a massive $\mathcal{N} = 4$ supermultiplet it is convenient to label the states $|j;\ell_L,\ell_R\rangle$ by the space-time helicity j and, in the second and third entry, by spin quantum numbers ℓ_L and ℓ_R of SU$(2)_L$ and SU$(2)_R$, respectively. As usual, we start from a 'Clifford vacuum' as the lowest state. For the smallest multiplets we choose

$$|\Omega\rangle \;=\; |j_0; 0, -\tfrac{1}{2}\rangle \;, \tag{41}$$

which is annihilated by all $a_{a\dot{b}}$. Assuming $M = m$, (40) implies that only $a^\dagger_{-\dot{+}}$ and $a^\dagger_{+\dot{+}}$ are active. Thus we obtain two states with helicity $j_0 + \frac{1}{2} : |j_0 + \frac{1}{2}; \frac{1}{2}, 0\rangle$ and $|j_0 + \frac{1}{2}; -\frac{1}{2}, 0\rangle$. Due to (40) and the anticommutativity of the oscillators, acting on $|j_0 + \frac{1}{2}; \frac{1}{2}, 0\rangle$ only the operator $a^\dagger_{+\dot{-}}$ can be potentially non-zero. However, one finds

$$a^\dagger_{+\dot{-}}|j_0 + \tfrac{1}{2}; \tfrac{1}{2}, 0\rangle \;=\; a^\dagger_{+\dot{-}}a^\dagger_{+\dot{+}}|\Omega\rangle \;=\; -a^\dagger_{+\dot{+}}a^\dagger_{+\dot{-}}|\Omega\rangle \;=\; 0 \;, \tag{42}$$

where we used in the last equation that $a^{\dagger}_{+\dot{-}}$ is inactive on the vacuum $|\Omega\rangle$. Similarly, one derives that there are no other states of helicity higher than $j_0 + \frac{1}{2}$. Moreover, by acting with the $\mathrm{SU}(2)$ raising and lowering operators M_{++}, etc., the states combine into complete $\mathrm{SU}(2)$ representations.

Summarizing, the action of the creation operators acting on the vacuum raises the helicity from j_0 to $j_0 + \frac{1}{2}$ and converts the complex $2_{\mathbb{C}}$ doublet $\mathrm{SU}(2)_R$ representation of the ground state into a complex doublet $\mathrm{SU}(2)_L$ representation of the next $j_0 + \frac{1}{2}$ state which we indicate by $\bar{2}_{\mathbb{C}}$. Note that, for $j_0 = -\frac{1}{2}$, there is no state with space-time helicity $+\frac{1}{2}$ and therefore the corresponding scalar multiplet is parity-odd.

This pattern will repeat itself in the $\mathcal{N} = 4k$ cases. For each of the k $\mathrm{SO}(4)$ factors one repeats the same calculation as above. Each $\mathrm{SO}(4)$ factor is written as the product of a $\mathrm{SU}(2)_L$ times a $\mathrm{SU}(2)_R$. One starts from a ground state $|\Omega\rangle$ which is in the $(2, 2, \cdots, 2)$ (k factors of 2) representation of all the $\mathrm{SU}(2)_R$ factors. Note that this representation is complex for k odd but can be taken to be real for k even. The first excited state, with helicity $j_0 + \frac{1}{2}$ is obtained by replacing one of the 2 representations by a $\bar{2}$. This can be done in k different ways. The next excited state is obtained by replacing in $|\Omega\rangle$ two 2 representations by $\bar{2}$ which can be done in $\binom{k}{2}$ different ways, etc. For the convenience of the reader we have summarized the structure of the short multiplets for $\mathcal{N} = 4, 8$ and 16 in Table 3.

helicity	$\mathcal{N} = 4$	$\mathcal{N} = 8$	$\mathcal{N} = 16$
j_0	$2_{\mathbb{C}}$	(2,2)	(2,2,2,2)
$j_0 + \frac{1}{2}$	$\bar{2}_{\mathbb{C}}$	$(\bar{2}, 2) + (2, \bar{2})$	$(\bar{2}, 2, 2, 2)$ + 3 more
$j_0 + 1$		$(\bar{2}, \bar{2})$	$(\bar{2}, \bar{2}, 2, 2)$ + 5 more
$j_0 + \frac{3}{2}$			$(\bar{2}, \bar{2}\bar{2}, 2)$ + 3 more
$j_0 + 2$			$(\bar{2}, \bar{2}, \bar{2}, \bar{2})$
d.o.f.	$4_B + 4_F$	$8_B + 8_F$	$128_B + 128_F$

Table 3 Multiplet structure for different values of $\mathcal{N}$, containing the space-time helicity j, the (real and complex) representations of the broken R-symmetry group and the total number of d.o.f.

Remarkably, there exists a vector multiplet with $\mathcal{N} = 16$ supersymmetry. According to Table 3, with $j_0 = -1$, it contains 96 massive helicity 0 states and 32 massive states with helicity ± 1. In fact, this symmetry is realized in the spectrum of fluctuations around the maximally supersymmetric Minkowski vacuum of the $\mathcal{N} = 16$ $\mathrm{SO}(4,4)^2$ gauged supergravity in three dimensions [23]. This vacuum breaks the gauge group G_0 down to its maximal compact subgroup $\mathrm{SO}(4)^4$ leading to 32 broken symmetries. This precisely leads to 32 massive spin 1 states in the fluctuation spectrum.

5 Outlook

In this talk we discussed the construction of superconformal gaugings in three dimensions with $\mathcal{N} = 5, 6$ and 8 supersymmetry. Furthermore we pointed out that the massive deformations of these models realize a supersymmetry algebra with a non-central extension. We showed how a massive $\mathcal{N} = 16$ massive super-multiplet could be constructed. This multiplet is realized in the context of maximal gauged supergravity with gauge group $\mathrm{SO}(4,4)^2$.

Independent of its applications to the theory of multiple M2-branes it is of interest to investigate whether interacting superconformal theories can be constructed for $\mathcal{N} > 8$. We already discussed the massive (and hence non-conformal) deformations but these only lead to free field theories. To introduce interactions one must consider gaugings. Interestingly, there are $\mathcal{N} > 8$ gauged supergravities with a $\hat{G}/\hat{H}$ coset structure in which $\hat{H}$ is given by the product of the R-symmetry group and the non-Abelian group $\mathrm{SU}(2)$. This suggests that non-trivial $\mathrm{SU}(2)$ gaugings could exist. Unfortunately, we have not been able to construct such gaugings. Therefore, the conclusion seems to be that no interacting theories can be constructed beyond $\mathcal{N} = 8$.

Finally, it is of interest to also consider the superconformal gaugings with $\mathcal{N} = 1, 2$ and 4 supersymmetries. We already mentioned the work of [13] which discusses the $\mathcal{N} = 4$ case and its relation to Lie superalgebras. The special thing about $\mathcal{N} = 1, 2$ and 4 supersymmetry is that the corresponding gauged supergravities are not anymore restricted to coset manifolds. This leads to more possibilities for constructing superconformal gaugings with possibly non-trivial worldvolumes and/or target spaces. These cases are presently under study [24].

Acknowledgements We wish to thank our collaborators M. de Roo, D. Roest, H. Samtleben and E. Sezgin. The work presented in this talk was based on collaborations with them. We furthermore would like to thank the organizers of the conference for a very stimulating meeting organized in such an inspiring atmosphere.

References

[1] E. A. Bershoeff, E. Sezgin, and P. K Townsend, Phys. Lett. B **189** 75 (1987).
[2] J. H. Schwarz, JHEP, J. High Energy Phys. **0411** 078 (2004).
[3] J. Bagger and N. Lambert, Phys. Rev. D **75** 045020 (2007).
[4] J. Bagger and N. Lambert, Phys. Rev. D **77** 065008 (2008).
[5] J. Bagger and N. Lambert, JHEP, J. High Energy Phys. **0802** 105 (2008).
[6] A. Gustavsson, arXiv:0709.1260 [hep-th].
[7] A. Gustavsson, JHEP, J. High Energy Phys. **0804** 083 (2008).
[8] E. A. Bergshoeff, O. Hohm, D. Roest, H. Samtleben, and E. Sezgin, JHEP, J. High Energy Phys. **0809** 101 (2008).
[9] E. A. Bergshoeff and O. Hohm, Phys. Rev. D **78** 125017 (2008).
[10] H. Nicolai and H. Samtleben, Phys. Rev. Lett. **86** 1686 (2001).
[11] H. Nicolai and H. Samtleben, JHEP, J. High Energy Phys. **0104** 022 (2001).
[12] B. de Wit, I. Herger, and H. Samtleben, Nucl. Phys. B **671** 175 (2003).
[13] D. Gaiotto and E. Witten, arXiv:0804.2907 [hep-th].
[14] K. Hosomichi, K. M. Lee, S. Lee, S. Lee, and J. Park, JEP **0809** 002 (2008).
[15] J. Bagger and N. Lambert, Phys. Rev. D **79** 025002 (2009).
[16] E. A. Bergshoeff, M. de Roo, O. Hohm, and D. Roest, JHEP, J. High Energy Phys. **0808** 091 (2008).
[17] M. Schnabl and Y. Tachikawa, arXiv:0807.1102 [hep-th].
[18] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, JHEP, J. High Energy Phys. **0810** 091 (2008).
[19] E. A. Bergshoeff, M. de Roo, and O. Hohm, Class. Quant. Grav. **25** 142001 (2008).
[20] J. Gomis, A. J. Salim, and F. Passerini, JHEP, J. High Energy Phys. **0808** 002 (2008).
[21] K. Hosomichi, K. M. Lee, and S. Lee, Phys. Rev. D **78** 066015 (2008).
[22] W. Nahm, Nucl. Phys. B **135** 149 (1978).
[23] T. Fischbacher, H. Nicolai, and H. Samtleben, Class. Quantum Gravity **19** 5297 (2002).
[24] E. A. Bergshoeff, S. Cecotti, H. Samtleben, and E. Sezgin, work in progress.